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Conformal Invariance and Conserved Quantity for the Nonholonomic System of Chetaev's Type

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Abstract In this paper the definition of conformal invariance and determining equation for the holonomic system which correspond to a nonholonomic system of Chetaev's type are provided. Conformal factor expression is deduced through relationship between a system's conformal invariance and Lie symmetry. The necessary and sufficient condition that the system's conformal invariance would be Lie symmetry under transformations by the infinitesimal one-parameter transformation group is obtained. The conformal invariance of weak and strong Lie symmetry for the nonholonomic system of Chetaev's type is given using restriction equations and additional restriction equations. And the system's corresponding conserved quantity is derived with the aid of a structure equation that gauge function satisfied. Lastly, an example is taken to illustrate the application of the result.

Keywords Nonholonomic system of Chetaev's type · Conformal invariance · Lie symmetry · Conserved quantity

1 Introduction

Since Noether theorem [1, 2] was published in 1918, the modern symmetrical theory has not only been used for finding a few conserved quantities that classical analytical mechanics has already revealed, such as the generalizations of conservation of momentum and conservation of mechanical energy and so on, but also found more conserved quantities. Even though differential equations of motion for a system are nonintegrable, the existence of some conserved quantities can enable us to get some knowledge about the local physical state. In recent years, more and more attention has been paid to research on symmetries and conserved quantities for dynamical systems, and some important results have been obtained so far [3–16].

Conformal invariance is built on the scale invariance, translation invariance, rotational invariance and a variety of interactions. The method is of great theoretical significance and

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application value. Galiullin et al. have studied conformal invariance, or conformal symmetry, of Birkhoff system under special infinitesimal transformations [17]. Recently, we have discussed the conformal invariance and conserved quantities of Lie symmetry or Mei symmetry for Lagrange systems, Hamilton systems and general holonomic mechanical systems [18–22]. The authors of Refs. [23] and [24] have conducted conformal invariance of first-order differential equations and generalized conformal symmetries of Hamilton system, respectively.

As dynamical equations of a nonholonomic system is much more complex than those of a holonomic system, the research of their symmetries and conserved quantities must be more difficult. The theory of symmetries and conserved quantities for general nonholonomic systems can be applied to holonomic systems as well. However, till now, the research on conformal invariance for nonholonomic systems has not been reported yet. In this work, we study conformal invariance and conserved quantities for the nonholonomic system of Chetaev's type.

2 Differential Equations of Motion for the Constrained System

Suppose that the configuration of a mechanical system is determined by n generalized coordinates q_s $(s = 1, 2, \dots, n)$. The system is subject to g ideal two-sided Chetaev's-type nonholonomic constraints

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (\beta = 1, \dots, g). \tag{1}$$

Chetaev conditions of nonholonomic constraints added to the virtual displacement δq_s are

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0, \tag{2}$$

differential equations of motion for the nonholonomic system of Chetaev's type are written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s = 1, \dots, n), \tag{3}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is Lagrangian of the system, $Q_j = Q_j(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are non-potential generalized forces, λ_β are constraint multipliers. λ_β can be expressed as explicit functions of variables $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$ before getting the integral of differential equations of motion. Thus (3) can be expressed as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s, \tag{4}$$

where $\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\frac{\partial f_\beta}{\partial \dot{q}_s}$ are constraint forces of the nonholonomic system. Equation (4) is called differential equation of motion for the holonomic system which correspond to the nonholonomic system in (1) and (3).

From (4) we can get

$$F_s \equiv A_{sk}(t, \boldsymbol{q})\ddot{q}_k + B_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \tag{5}$$

where

$$A_{sk} = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}, \qquad B_s = \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial L}{\partial q_s}. \tag{6}$$

Suppose that the system is not singular, i.e. $D=\det(\frac{\partial^2 L}{\partial\dot q_s\partial\dot q_k})\neq 0$, using (5), we may get all generalized accelerations

$$\ddot q_s=\frac{M_{sk}}{D}\left(\frac{\partial L}{\partial q_k}-\frac{\partial^2 L}{\partial\dot q_k\partial t}-\frac{\partial^2 L}{\partial\dot q_k\partial q_j}\dot q_j+Q_k+\Lambda_k\right), \tag{7}$$

where M_{sk} are the cofactors of the matrix element $\partial^2 L/\partial\dot q_s\partial\dot q_k$. Equation (7) can be rewritten as

$$\ddot q_s=\alpha_s(t,\boldsymbol q,\dot{\boldsymbol q})\quad (s=1,\dots,n). \tag{8}$$

3 Conformal Invariance and Its Determining Equation

We introduce a one-parameter infinitesimal transformation with respect to time and coordinates

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol q,\dot{\boldsymbol q}),\qquad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol q,\dot{\boldsymbol q}), \tag{9}$$

where ε is a infinitesimal transformation parameter, and $\xi_0,\ \xi_s$ are infinitesimal generators.

Taking the infinitesimal generator vector

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{10}$$

its first prolonged vector is

$$X^{(1)}=X^{(0)}+(\dot\xi_s-\dot q_s\dot\xi_0)\frac{\partial}{\partial\dot q_s}, \tag{11}$$

and its second prolonged vector is

$$X^{(2)}=X^{(1)}+(\ddot\xi_s-2\ddot q_s\dot\xi_0-\dot q_s\ddot\xi_0)\frac{\partial}{\partial\ddot q_s}. \tag{12}$$

Definition 1 If there is a matrix Γ_s^l satisfying the following conditions

$$X^{(2)}(F_s)=\Gamma_s^l(F_l)\quad (s,l=1,\dots,n) \tag{13}$$

then this kind of invariance is called the conformal invariance of (5) under the one-parameter infinitesimal transformations in (9). Equations (13) are determining equations of conformal invariance for (5), and Γ_s^l is called its conformal factor.

4 Conformal invariance and Lie symmetry

Definition 2 If infinitesimal generators $\xi_0,\ \xi_s$ satisfy determining equations

$$X^{(2)}(F_s)|_{F_s=0}=0, \tag{14}$$

then this kind of symmetry is called the Lie symmetry for the holonomic system in (5) which corresponds to the nonholonomic system in (1) and (3).

The invariance of nonholonomic constraint equations (1), under the infinitesimal transformations in (9), gives the following restriction equations

$$X^{(1)}[f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})]=0 \quad (\beta=1,\dots,g). \tag{15}$$

Definition 3 If infinitesimal generators $\xi_0,\ \xi_s$ satisfy the determining equations (14) and restriction equations (15), then the symmetry is called weak Lie symmetry for the nonholonomic system.

Considering the restriction of the Chetaev condition (2) on infinitesimal generators $\xi_0,\ \xi_s$, we can have

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s-\dot{q}_s\xi_0)=0. \tag{16}$$

Equation (16) is called additional restriction equation.

Definition 4 If infinitesimal generators $\xi_0,\ \xi_s$ satisfy determining equations (14), restriction equations (15) and additional restriction equations (16), then the symmetry is called strong Lie symmetry for the nonholonomic system.

Theorem 1 *For* (5), *if generators ξ_0 and ξ_s for infinitesimal transformations* (9) *satisfy Lie symmetry, and there is a matrix β_s^l satisfying the following condition*

$$X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}=\beta_s^l F_l \quad (s,l=1,\dots,n), \tag{17}$$

then the necessary and sufficient condition that the conformal invariance would be Lie symmetry is

$$\Gamma_s^l=\beta_s^l. \tag{18}$$

Proof In fact, the Lie symmetry for (5) satisfy

$$X^{(2)}(F_s)|_{F_s=0}=0, \tag{19}$$

if there is a matrix β_s^l satisfying (17), then it becomes

$$X^{(2)}(F_s)=\beta_s^l(F_l), \tag{20}$$

according to definition (13),the conformal factor of the system is $\Gamma_s^l=\beta_s^l$.

Vice versa, according to definition (13) and (17), we can easily verify that

$$(\Gamma_s^l-\beta_s^l)(F_l)=X^{(2)}(F_s)|_{F_s=0} \quad (s,l=1,\dots,n). \tag{21}$$

If $\Gamma_s^l=\beta_s^l$, then we have $X^{(2)}(F_s)|_{F_s=0}=0$, the system is Lie symmetrical. □

5 Conformal Factor

To obtain the conformal factor of conformal invariance for (5), let's take

$$X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}. \tag{22}$$

Using

$$\dot{\xi}_k = \frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial q_r}\dot{q}_r + \frac{\partial \xi_k}{\partial \dot{q}_r}\ddot{q}_r \quad (k = 0, 1, \ldots, n) \tag{23}$$

$$\begin{aligned}\ddot{\xi}_k &= \frac{\partial^2 \xi_k}{\partial t^2} + 2\frac{\partial^2 \xi_k}{\partial q_r \partial t}\dot{q}_r + 2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial q_r \partial q_j}\dot{q}_r\dot{q}_j + 2\frac{\partial^2 \xi_k}{\partial q_r \partial \dot{q}_j}\dot{q}_r\ddot{q}_j \\ &\quad + \frac{\partial \xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\ddot{q}_j + \frac{\partial \xi_k}{\partial \dot{q}_r}\left(\frac{\partial \alpha_r}{\partial t} + \frac{\partial \alpha_r}{\partial q_j}\dot{q}_j + \frac{\partial \alpha_r}{\partial \dot{q}_j}\ddot{q}_j\right) \\ &\quad (k = 0, 1, \ldots, n;\ r, j = 1, \ldots, n),\end{aligned} \tag{24}$$

we have

$$\begin{aligned}X^{(2)}(F_s) &= A_{sk}(\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0) + X^{(0)}(A_{sk})\ddot{q}_k \\ &\quad + X^{(0)}(B_s - Q_s - \Lambda_s) + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial(B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k} \\ &= A_{sk}\left[2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\ddot{q}_r + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j\ddot{q}_r + \frac{\partial \xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\ddot{q}_j \right. \\ &\quad \left. + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\ddot{q}_r - 2\ddot{q}_k\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r + \frac{\partial \xi_0}{\partial \dot{q}_r}\ddot{q}_r\right)\right] \\ &\quad - A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t}\ddot{q}_r + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j\ddot{q}_r + \frac{\partial \xi_0}{\partial q_r}\ddot{q}_r + \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\ddot{q}_j + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\ddot{q}_r\right) \\ &\quad + X^{(0)}(A_{sk})\ddot{q}_k + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k\frac{\partial \xi_0}{\partial \dot{q}_r}\right)\ddot{q}_r\frac{\partial(B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k} + C(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\end{aligned} \tag{25}$$

where $C(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the sum of terms which don't have $\ddot{q}_s$. By the same reasoning,

$$\begin{aligned}X^{(2)}(F_s)|_{F_s=0} &= A_{sk}\left[2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\alpha_r + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j\alpha_r + \frac{\partial \xi_k}{\partial q_r}\alpha_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\alpha_r\alpha_j \right. \\ &\quad \left. + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\alpha_r - 2\alpha_k\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r + \frac{\partial \xi_0}{\partial \dot{q}_r}\alpha_r\right)\right] \\ &\quad - A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t}\alpha_r + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j\alpha_r + \frac{\partial \xi_0}{\partial q_r}\alpha_r + \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}\alpha_r\alpha_j \right. \\ &\quad \left. + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\alpha_r\right) + X^{(0)}(A_{sk})\alpha_k + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k\frac{\partial \xi_0}{\partial \dot{q}_r}\right) \\ &\quad \times \alpha_r\frac{\partial(B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k} + C(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\end{aligned} \tag{26}$$

where $\alpha_s = -A^{sk}(B_k - Q_k - \Lambda_k)$. Let (25) subtract (26), we obtain

$$\begin{aligned}&X^{(2)}(F_s) - X^{(2)}(F_s)|_{F_s=0} \\ &\quad = A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r) + A_{sk}\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}(\ddot{q}_r\ddot{q}_j - \alpha_r\alpha_j)\end{aligned}$$

$$- 2A_{sk}(\ddot{q}_k - \alpha_k)\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) - 2A_{sk}\frac{\partial \xi_0}{\partial \dot{q}_r}(\ddot{q}_k\ddot{q}_r - \alpha_k\alpha_r)$$
$$- A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r)$$
$$- A_{sk}\dot{q}_k\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}(\ddot{q}_r\ddot{q}_j - \alpha_r\alpha_j) + X^{(0)}(A_{sk})(\ddot{q}_k - \alpha_k) + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k\frac{\partial \xi_0}{\partial \dot{q}_r}\right)$$
$$\times (\ddot{q}_r - \alpha_r)\frac{\partial(B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k} \quad (s, k, r, j = 1, \dots, n). \tag{27}$$

As

$$\ddot{q}_k - \alpha_k = \ddot{q}_k + A^{kl}(B_l - Q_l - \Lambda_l)$$
$$= A^{kl}(A_{lm}\ddot{q}_m + B_l - Q_l - \Lambda_l) = A^{kl}F_l, \tag{28}$$

$$\ddot{q}_k\ddot{q}_j = (A^{kl}F_l + \alpha_k)(A^{jm}F_m + \alpha_j)$$
$$= A^{kl}F_l A^{jm}F_m + \alpha_k A^{jm}F_m + \alpha_j A^{kl}F_l + \alpha_k\alpha_j, \tag{29}$$

therefore

$$X^{(2)}(F_s) - X^{(2)}(F_s)|_{F_s=0}$$
$$= \left[A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)A^{rl} - 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)\right.$$
$$-A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)A^{rl}$$
$$+ X^{(0)}(A_{sk})A^{kl} + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k\frac{\partial \xi_0}{\partial \dot{q}_r}\right)A^{rl}\frac{\partial(B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k}$$
$$+ A_{sk}\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_r A^{jl} + \alpha_j A^{rl}) - 2A_{sk}\frac{\partial \xi_0}{\partial \dot{q}_r}(\alpha_k A^{rl} + \alpha_r A^{kl})$$
$$\left.- A_{sk}\dot{q}_k\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_r A^{jl} + \alpha_j A^{rl})\right]F_l$$
$$+ A_{sk}\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}A^{jm}F_m A^{rl}F_l - 2A_{sk}\frac{\partial \xi_0}{\partial \dot{q}_r}A^{rm}F_m A^{kl}F_l - A_{sk}\dot{q}_k\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}A^{jm}F_m A^{rl}F_l$$
$$(s, k, r, j, m, l = 1, \dots, n). \tag{30}$$

Ignoring high-order terms of F_l, we have

$$X^{(2)}(F_s) - X^{(2)}(F_s)|_{F_s=0}$$
$$= \left[A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)A^{rl} - 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right)\right.$$
$$- A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)A^{rl}$$

$$+ X^{(0)}(A_{sk})A^{kl} + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k \frac{\partial \xi_0}{\partial \dot{q}_r}\right) A^{rl} \frac{\partial (B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k}$$

$$+ A_{sk} \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_r A^{jl} + \alpha_j A^{rl}) - 2A_{sk} \frac{\partial \xi_0}{\partial \dot{q}_r}(\alpha_k A^{rl} + \alpha_r A^{kl})$$

$$- A_{sk}\dot{q}_k \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_r A^{jl} + \alpha_j A^{rl})\Big] F_l$$

$$= \beta_s^l F_l \quad (s, k, r, j, l = 1, \ldots, n) \tag{31}$$

and we can find the conformal factors

$$\beta_s^l = A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right) A^{rl}$$

$$- A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_r}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right) A^{rl}$$

$$- 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) + X^{(0)}(A_{sk})A^{kl}$$

$$+ \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \dot{q}_k \frac{\partial \xi_0}{\partial \dot{q}_r}\right) A^{rl} \frac{\partial (B_s - Q_s - \Lambda_s)}{\partial \dot{q}_k}$$

$$+ A_{sk} \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_r A^{jl} + \alpha_j A^{rl}) - 2A_{sk} \frac{\partial \xi_0}{\partial \dot{q}_r}(\alpha_k A^{rl} + \alpha_r A^{kl})$$

$$- A_{sk}\dot{q}_k \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}(\alpha_r A^{jl} + \alpha_j A^{rl}) \quad (s, k, r, j, l = 1, \ldots, n). \tag{32}$$

If (5) can be formalized to their standard form

$$F_s \equiv \ddot{q}_s - \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \quad (s = 1, \ldots, n), \tag{33}$$

then their conformal factors are

$$\beta_s^l = 2\frac{\partial^2 \xi_s}{\partial \dot{q}_l \partial t} + 2\frac{\partial^2 \xi_s}{\partial q_j \partial \dot{q}_l}\dot{q}_j + \frac{\partial \xi_s}{\partial q_l} + \frac{\partial \xi_s}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_l}$$

$$- \dot{q}_s\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_l \partial t} + 2\frac{\partial^2 \xi_0}{\partial q_j \partial \dot{q}_l}\dot{q}_j + \frac{\partial \xi_0}{\partial q_l} + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_l}\right)$$

$$- 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r + \frac{\partial \xi_0}{\partial \dot{q}_r}\alpha_r\right) - \left(\frac{\partial \xi_k}{\partial \dot{q}_l} - \dot{q}_k \frac{\partial \xi_0}{\partial \dot{q}_l}\right)\frac{\partial \alpha_s}{\partial \dot{q}_k}$$

$$+ 2\frac{\partial^2 \xi_s}{\partial \dot{q}_r \partial \dot{q}_l}\alpha_r - 2\dot{q}_s \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_l}\alpha_r - 2\frac{\partial \xi_0}{\partial \dot{q}_l}\alpha_s. \tag{34}$$

In fact, now (34) is equal to $A_{sk} = \delta_{sk}$, $B_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) - \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = -\alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ in (5). From (32), (34) can be easily obtained.

If infinitesimal generators ξ_0, ξ_s satisfy the determining equations (14) and restriction equations (15), then aforementioned conformal invariance is the conformal invariance of weak Lie symmetry for the nonholonomic system of Chetaev's type.

If infinitesimal generators $\xi_0,\ \xi_s$ satisfy the determining equations (14), restriction equations (15) and additional restriction equations (16), then aforementioned conformal invariance is the conformal invariance of strong Lie symmetry for the nonholonomic system of Chetaev's type.

6 Structure Equation and Conservation Quantity

The conformal invariance can also lead to a conserved quantity under certain conditions.

Theorem 2 *If generators ξ_0 and ξ_s of infinitesimal transformations* (9) *satisfy the conformal factor* (32) *or* (34), *and there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following Lie symmetrical structure equation*

$$L\dot{\xi}_0 + X^{(1)}(L) + (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G} = 0, \tag{35}$$

then the conformal invariance of holonomic system (5) *which corresponds to the nonholonomic system of Chetaev's type in* (1) *and* (3) *possesses the following conserved quantity*

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G = \text{const.} \tag{36}$$

Proof

$$\begin{aligned}\frac{dI}{dt} &= \dot{L}\xi_0 + L\dot{\xi}_0 + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \ddot{q}_s\xi_0 - \dot{q}_s\dot{\xi}_0)\\ &\quad - L\dot{\xi}_0 - \frac{\partial L}{\partial t}\xi_0 - \frac{\partial L}{\partial q_s}\xi_s - \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) - (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0)\\ &= (\xi_s - \dot{q}_s\xi_0)\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s - \Lambda_s\right) = 0.\end{aligned} \tag{37}$$

□

Theorem 3 *If $\xi_0,\ \xi_s$ are generators of weak* (*strong*) *Lie symmetry for the nonholonomic system of Chetaev's type in* (1) *and* (3), *and there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structural equation* (35), *then the conformal invariance of weak* (*strong*) *Lie symmetry for the nonholonomic system of Chetaev's type in* (1) *and* (3) *possesses the conserved quantity* (36).

7 An Illustrative Example

A nonholonomic system of Chetaev's type is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - q_2, \tag{38}$$

$$f = \dot{q}_1 - t\dot{q}_2 = 0, \tag{39}$$

$$Q_1 = 0, \qquad Q_2 = -2\dot{q}_1, \tag{40}$$

its differential equations of motion is

$$\begin{aligned} \ddot{q}_1 &= \lambda, \\ \ddot{q}_2 &= -1 - \lambda t - 2\dot{q}_1, \end{aligned} \tag{41}$$

from (38)–(41), we can get

$$\lambda = \frac{1}{1+t^2}(\dot{q}_2 - 2t\dot{q}_1 - t), \tag{42}$$

therefore

$$\begin{aligned} \Lambda_1 &= \lambda = \frac{1}{1+t^2}(\dot{q}_2 - 2t\dot{q}_1 - t), \\ \Lambda_2 &= -\lambda t = -\frac{t}{1+t^2}(\dot{q}_2 - 2t\dot{q}_1 - t), \end{aligned} \tag{43}$$

$$\begin{aligned} \ddot{q}_1 &= \frac{1}{1+t^2}(\dot{q}_2 - 2t\dot{q}_1 - t), \\ \ddot{q}_2 &= -\frac{1}{1+t^2}(1 + 2\dot{q}_1 + t\dot{q}_2), \end{aligned} \tag{44}$$

or

$$\begin{aligned} F_1 &= \ddot{q}_1 - \frac{1}{1+t^2}(\dot{q}_2 - 2t\dot{q}_1 - t), \\ F_2 &= \ddot{q}_2 + \frac{1}{1+t^2}(1 + 2\dot{q}_1 + t\dot{q}_2). \end{aligned} \tag{45}$$

Taking

$$\xi_0 = 0, \qquad \xi_1 = 0, \qquad \xi_2 = t\dot{q}_1 + \dot{q}_2 + q_1 + t, \tag{46}$$

we have

$$\begin{aligned} X^{(2)}(F_1) &= -(1 + 2\dot{q}_1 + t\ddot{q}_1 + \ddot{q}_2)/(1+t^2) \\ &= -t/(1+t^2)F_1 - 1/(1+t^2)F_2, \\ X^{(2)}(F_2) &= (2t^2\ddot{q}_1 + 2t + 4\dot{q}_1 t + \ddot{q}_2 t - \dot{q}_2 + \ddot{q}_1)/(1+t^2) \\ &= (2t^2+1)/(1+t^2)F_1 + t/(1+t^2)F_2. \end{aligned} \tag{47}$$

Therefore,

$$\Gamma = \begin{pmatrix} -t/(1+t^2) & -1/(1+t^2) \\ (2t^2+1)/(1+t^2) & t/(1+t^2) \end{pmatrix}. \tag{48}$$

From (34), we can also obtain the conformal factor $\beta_s^k = \Gamma_s^k$. Then the conformal invariance of the system would be Lie symmetry.

Apparently, the generators satisfy the following restriction equation

$$X^{(1)} f = X^{(1)}(\dot{q}_1 - t\dot{q}_2) = 0, \tag{49}$$

therefore, the corresponding symmetry is the weak Lie symmetry for the system. But the generator does not meet the additional restriction equations (16), i.e.

$$\frac{\partial f}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) \neq 0 \quad (s = 1, 2) \tag{50}$$

therefore, the corresponding symmetry is not a strong Lie symmetry for the system.

Substitute (38), (40), (43) and (46) into structural equation (35), we can obtain

$$\dot{G} = (1 + 2\dot{q}_1 + t\dot{q}_2)(t\dot{q}_1 + \dot{q}_2 + q_1 + t)/(1 + t^2), \tag{51}$$

therefore

$$G = (t\dot{q}_1 + \dot{q}_2 + q_1 + t)(t\dot{q}_1 + q_1 + t). \tag{52}$$

The conserved quantity (36) gives

$$I = (t\dot{q}_1 + \dot{q}_2 + q_1 + t)^2 = \text{const}. \tag{53}$$

8 Conclusion

For a holonomic system which corresponds to a nonholonomic system of Chetaev's type, by using the definition of conformal invariance and Lie symmetrical determining equations, we can obtain the conformal factor of determining equations of conformal invariance. The conformal factor is also the necessary and sufficient condition that the conformal invariance would be Lie symmetry of the system. If infinitesimal generators satisfy restriction equations, aforementioned conformal invariance is the conformal invariance of weak Lie symmetry for the nonholonomic system of Chetaev's type. If generators satisfy restriction equations and additional restriction equations, it is the conformal invariance of strong Lie symmetry. The conformal invariance can also lead to a conserved quantity under certain condition.

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